Topology and its Applications 156 (2009) 601–609

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

One-dimensional locally connected S-spaces

Joan E. Hart [a,*,1], Kenneth Kunen [b,1]

[a] University of Wisconsin, Oshkosh, WI 54901, USA
[b] University of Wisconsin, Madison, WI 53706, USA

ARTICLE INFO

Article history:
Received 27 March 2008
Received in revised form 22 July 2008
Accepted 20 August 2008

MSC:
primary 54D05, 54D65

Keywords:
One-dimensional
Peano continuum
Locally connected
Convergent sequence
Menger curve
S-space

ABSTRACT

We construct, assuming Jensen's principle $\diamondsuit$, a one-dimensional locally connected hereditarily separable continuum without convergent sequences.

1. Introduction

All topologies discussed in this paper are assumed to be Hausdorff. A *continuum* is any compact connected space. A *nontrivial convergent sequence* is a convergent ω-sequence of distinct points. As usual, $\dim(X)$ is the covering dimension of X; for details, see Engelking [7]. "HS" abbreviates "hereditarily separable". We shall prove:

Theorem 1.1. *Assuming $\diamondsuit$, there is a locally connected HS continuum Z such that $\dim(Z) = 1$ and Z has no nontrivial convergent sequences.*

Note that points in Z must have uncountable character, so that Z is not hereditarily Lindelöf; thus, Z is an S-space.

Spaces with some of these features are well known from the literature. A compact F-space has no nontrivial convergent sequences. Such a space can be a continuum; for example, the Čech remainder $\beta[0,1) \setminus [0,1)$ is connected, although not locally connected; more generally, no infinite compact F-space can be either locally connected or HS. In [15], van Mill constructs, under the Continuum Hypothesis, a locally connected continuum with no nontrivial convergent sequences. Van Mill's example, constructed as an inverse limit of Hilbert cubes, is infinite dimensional. Here, we shall replace the Hilbert cubes by one-dimensional Peano continua (i.e., connected, locally connected, compact metric spaces) to obtain a one-dimensional limit space. Our $Z = Z_{\omega_1}$ will be the limit of an inverse system $\langle Z_\alpha \colon \alpha < \omega_1 \rangle$. Each Z_α will be a copy of the *Menger*

* Corresponding author.
E-mail addresses: hartj@uwosh.edu (J.E. Hart), kunen@math.wisc.edu (K. Kunen).
[1] Author partially supported by NSF Grant DMS-0456653.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.08.009

sponge [13] (or Menger curve) MS; this one-dimensional Peano continuum has homogeneity properties similar to those of the Hilbert cube. The basic properties of MS are summarized in Section 2, and Theorem 1.1 is proved in Section 3.

In [15], as well as in earlier work by Fedorchuk [9] and van Douwen and Fleissner [4], one kills all possible nontrivial convergent sequences in ω_1 steps. Here, we focus primarily on obtaining an S-space, modifying the construction of the original Fedorchuk S-space [8]; we follow the exposition in [5], where the lack of convergent sequences occurs only as an afterthought. This exposition can easily be modified to make Z a *strong* S-space as well; see Section 5.

We do not know whether one can obtain Z so that it satisfies Theorem 1.1 with the stronger property $\mathrm{ind}(Z) = 1$; that is, the open $U \subseteq Z$ with ∂U zero-dimensional form a base. In fact, we can easily modify our construction to ensure that $1 = \dim(Z) < \mathrm{ind}(Z) = \infty$; this will hold because (as in [5]) we can give Z the additional property that all perfect subsets are G_δ sets; see Section 6 for details.

We can show that a Z satisfying Theorem 1.1 cannot have the property that the open $U \subseteq Z$ with ∂U scattered form a base; see Theorem 4.12 in Section 4. This strengthening of $\mathrm{ind}(Z) = 1$ is satisfied by some well-known Peano continua. It is also satisfied by the space produced in [10] under $\diamondsuit$ by a recursive construction related to the one we describe here, but the space of [10] was not locally connected, and it had nontrivial convergent sequences (in fact, it was hereditarily Lindelöf).

2. On sponges

The Menger sponge MS [13] is obtained by drilling holes through the cube $[0,1]^3$, analogously to the way that one obtains the middle-third Cantor set by removing intervals from $[0,1]$. The paper of Mayer, Oversteegen, and Tymchatyn [14] has a precise definition of MS and discusses its basic properties.

In proving theorems about MS, one often refers not to its definition, but to the following theorem of R.D. Anderson [1,2] (or, see [14]), which characterizes MS. This theorem will be used to verify inductively that $Z_\alpha \cong \mathrm{MS}$. The fact that MS satisfies the stated conditions is easily seen from its definition, but it is not trivial to prove that they characterize MS.

Theorem 2.1. MS *is, up to homeomorphism, the only one-dimensional Peano continuum with no locally separating points and no non-empty planar open sets.*

Here, $C \subseteq X$ is *locally separating* iff, for some connected open $U \subseteq X$, the set $U \setminus C$ is not connected. A point x is locally separating iff $\{x\}$ is. This notion is applied in the Homeomorphism Extension Theorem of Mayer, Oversteegen, and Tymchatyn [14]:

Theorem 2.2. *Let K and L be closed, non-locally-separating subsets of* MS *and let $h : K \twoheadrightarrow L$ be a homeomorphism. Then h extends to a homeomorphism of* MS *onto itself.*

The non-locally-separating sets have the following closure property of Kline [11] (or, see Theorem 2.2 of [14]):

Theorem 2.3. *Let X be compact and locally connected, and let $K = \bigcup\{K_i\colon i \in \omega\}$, where K and the K_i are closed subsets of X. If K is locally separating then some K_i is locally separating.*

For example, these results imply that in MS, all convergent sequences are equivalent. More precisely, points in MS are not locally separating, so if $\langle x_i\colon i \in \omega\rangle$ converges to x_ω, then $\{x_i\colon i \leqslant \omega\}$ is not locally separating. Thus, if $\langle s_i\rangle$ and $\langle t_i\rangle$ are nontrivial convergent sequences in MS, with limit points s_ω and t_ω, respectively, then there is a homeomorphism of MS onto itself that maps s_i to t_i for each $i \leqslant \omega$.

The following consequence of Theorem 2.1 was noted by Prajs [16] (see p. 657).

Lemma 2.4. *Let $J \subseteq \mathrm{MS}$ be a non-locally-separating arc and obtain MS/J by collapsing J to a point. Then $\mathrm{MS}/J \cong \mathrm{MS}$ and the natural map $\pi : \mathrm{MS} \twoheadrightarrow \mathrm{MS}/J$ is monotone.*

Here, a map $f : Y \twoheadrightarrow X$ is called *monotone* iff each $f^{-1}\{x\}$ is connected; so, the monotonicity in Lemma 2.4 is obvious. When X and Y are compact, monotonicity implies that $f^{-1}(U)$ is connected whenever U is a connected open or closed subset of X.

We shall use these results to show that the property of being a Menger sponge will be preserved at the limit stages of our construction:

Lemma 2.5. *Suppose that γ is a countable limit ordinal and Z_γ is an inverse limit of $\langle Z_\alpha\colon \alpha < \gamma\rangle$, where all bonding maps σ^β_α are monotone and each $Z_\alpha \cong \mathrm{MS}$. Then $Z_\gamma \cong \mathrm{MS}$.*

Proof. We verify the conditions of Theorem 2.1. $\dim(Z_\gamma) = 1$, since this property is preserved by inverse limits of compacta, and Z_γ is locally connected because the σ^β_α are monotone. So, we need to verify that Z_γ has no locally separating points and no non-empty planar open sets.

Suppose that $q \in Z_\gamma$ is locally separating; so we have a connected neighborhood U of q with $U \backslash \{q\}$ not connected. Shrinking U, we may assume that $U = (\sigma_\alpha^\gamma)^{-1}(V)$, where $\alpha < \gamma$ and V is open and connected in Z_α. Since $Z_\alpha \cong \mathsf{MS}$, $\sigma_\alpha^\gamma(q)$ is not locally separating, so $V \backslash \{\sigma_\alpha^\gamma(q)\}$ is connected. Then, since σ_α^γ is monotone, $(\sigma_\alpha^\gamma)^{-1}(V \backslash \{\sigma_\alpha^\gamma(q)\}) = U \setminus (\sigma_\alpha^\gamma)^{-1}\{\sigma_\alpha^\gamma(q)\}$ is connected. The same argument shows that $U \setminus (\sigma_\beta^\gamma)^{-1}\{\sigma_\beta^\gamma(q)\}$ is connected whenever $\alpha \leqslant \beta < \gamma$. But then $U \backslash \{q\} = \bigcup \{U \setminus (\sigma_\beta^\gamma)^{-1}\{\sigma_\beta^\gamma(q)\}\colon \alpha \leqslant \beta < \gamma\}$ is connected also.

Suppose that $U \subseteq Z_\gamma$ is open and non-empty; we show that U is not planar. Shrinking U, we may assume that $U = (\sigma_\alpha^\gamma)^{-1}(V)$, where $\alpha < \gamma$ and V is open in Z_α. Since $Z_\alpha \cong \mathsf{MS}$, there is a K_5 set $F \subseteq V$; that is, F consists of 5 distinct points p_0, p_1, p_2, p_3, p_4 together with arcs $J_{i,j}$ with endpoints p_i, p_j for $0 \leqslant i < j < 5$, where the sets $J_{i,j} \setminus \{p_i, p_j\}$, for $0 \leqslant i < j < 5$, are pairwise disjoint. Now F is not planar, and, one can show that $(\sigma_\alpha^\gamma)^{-1}(F)$ is not planar either. To do this, use the fact that σ_α^γ is monotone, so that the sets $(\sigma_\alpha^\gamma)^{-1}\{p_i\}$ and $(\sigma_\alpha^\gamma)^{-1}(J_{i,j})$ are all continua. □

The following terminology was used also in the exposition in [5] of the Fedorchuk S-space:

Definition 2.6. Let $\mathcal{F}$ be a family of subsets of X. Then $x \in X$ is a *strong limit point* of $\mathcal{F}$ iff for all neighborhoods U of x, there is an $F \in \mathcal{F}$ such that $F \subseteq U$ and $x \notin F$.

In practice, we shall only use this notion when the elements of $\mathcal{F}$ are closed. If all elements of $\mathcal{F}$ are singletons, this reduces to the usual notion of a point being a limit point of a set of points.

The map $\sigma_\alpha^{\alpha+1} : Z_{\alpha+1} \twoheadrightarrow Z_\alpha$ will always be obtained by collapsing a non-locally-separating arc in $Z_{\alpha+1}$ to a point. We obtain it using:

Lemma 2.7. *Assume that $X \cong \mathsf{MS}$ and that for $n \in \omega$, $\mathcal{F}_n$ is a family of non-locally-separating closed subsets of X. Fix $t \in X$ such that t is a strong limit point of each $\mathcal{F}_n$. Then there is a $Y \cong \mathsf{MS}$ and a monotone $\sigma : Y \twoheadrightarrow X$ such that*

1. *$\sigma^{-1}\{t\}$ is a non-locally-separating arc in Y,*
2. *$|\sigma^{-1}\{x\}| = 1$ for all $x \neq t$, and*
3. *y is a strong limit point of $\{\sigma^{-1}(F)\colon F \in \mathcal{F}_n\}$, for each $y \in \sigma^{-1}\{t\}$ and $n \in \omega$.*

Proof. First, let $\{A_n\colon n \in \omega\}$ partition ω into disjoint infinite sets. In X, choose disjoint closed $F_i \not\ni t$ for $i \in \omega$ such that $F_i \in \mathcal{F}_n$ whenever $i \in A_n$, and such that every neighborhood of t contains all but finitely many of the F_i. Let $L = \{t\} \cup \bigcup_i F_i$. Then L is closed and non-locally-separating by Theorem 2.3.

Now, in MS, let J be any non-locally-separating arc. Choose disjoint closed non-locally separating sets G_i for $i \in \omega$ such that each $G_i \cong F_i$, every neighborhood of J contains all but finitely many G_i, each $G_i \cap J = \emptyset$, and for each n and each $y \in J$, y is a strong limit point of $\{G_i\colon i \in A_n\}$.

Let $\rho : \mathsf{MS} \twoheadrightarrow \mathsf{MS}/J$ be the usual projection, and let $[J]$ denote the point to which ρ collapses the set J. Then $\mathsf{MS}/J \cong \mathsf{MS}$ by Lemma 2.4. In MS/J, let $K = \{[J]\} \cup \bigcup \{\rho(G_i)\colon i \in \omega\}$. Let $h : K \twoheadrightarrow L$ be a homeomorphism such that $h([J]) = t$ and each $h(\rho(G_i)) = F_i$. By Theorem 2.2, h extends to a homeomorphism $\widetilde{h} : \mathsf{MS}/J \twoheadrightarrow X$.

Now, let $Y = \mathsf{MS}$ and let $\sigma = \widetilde{h} \circ \rho$. □

The next lemma will simplify somewhat the description of our inverse limit:

Lemma 2.8. *In Lemma 2.7, we may obtain $Y \subseteq X \times [0,1]$, with $\sigma : Y \twoheadrightarrow X$ the natural projection.*

Proof. Start with any Y, σ, and t satisfying Lemma 2.7, and let $J := \sigma^{-1}\{t\}$. Apply the Tietze Extension Theorem to fix $f : Y \twoheadrightarrow [0,1]$ such that $f \restriction J : J \twoheadrightarrow [0,1]$ is a homeomorphism. Then $y \mapsto (\sigma(y), f(y))$ is one-to-one on Y, and hence $\widetilde{Y} := \{(\sigma(y), f(y))\colon y \in Y\} \subseteq X \times [0,1]$ satisfies Lemma 2.8. □

The following additional property of our σ will be useful:

Lemma 2.9. *Let t and $\sigma : Y \twoheadrightarrow X$ be as in Lemma 2.7 or 2.8. Assume that $H \subseteq X$ is closed and nowhere dense and not locally separating. Then $\sigma^{-1}(H) \subseteq Y$ is closed and nowhere dense and not locally separating.*

Proof. $\sigma^{-1}(H)$ is closed and nowhere dense because σ is continuous and irreducible. Also note that $\sigma^{-1}(H)$ is not locally separating if either $H = \{t\}$ (trivially) or $t \notin H$ (because σ is a homeomorphism in a neighborhood of $\sigma^{-1}(H)$).

Next, note that every closed $K \subseteq H$ is non-locally-separating in X: If not, let $U \subseteq X$ be connected and open with $U \backslash K$ not connected, so that $U \backslash K = W_0 \cup W_1$, where the W_i are open in X, non-empty, and disjoint. Then $U \backslash H = W_0 \backslash H \cup W_1 \backslash H$, but H is not locally separating, so one of the $W_i \backslash H = \emptyset$, so $W_i \subseteq H$, contradicting H being nowhere dense.

Now, let $H = \bigcup_{n \in \omega} K_n$, where each K_n is closed and either $K_n = \{t\}$ or $t \notin K_n$. Then $\sigma^{-1}(H) = \bigcup_n \sigma^{-1}(K_n)$, which is not locally separating by Theorem 2.3. □

3. The inverse limit

We shall obtain our space $Z = Z_{\omega_1}$ as an inverse limit of a sequence $\langle Z_\alpha\colon \alpha < \omega_1\rangle$. As with many such constructions, it is somewhat simpler to view the Z_α concretely as subsets of cubes, so that the bonding maps are just projections. Thus, we shall have:

Conditions 3.1. We obtain Z_α for $\alpha \leqslant \omega_1$ and $\pi^\beta_\alpha, \sigma^\beta_\alpha$ for $\alpha \leqslant \beta \leqslant \omega_1$ such that:

(C1) Each Z_α is a closed subset of $\mathsf{MS} \times [0,1]^\alpha$, and $Z_0 = \mathsf{MS}$.
(C2) For $\alpha \leqslant \beta \leqslant \omega_1$, $\pi^\beta_\alpha : \mathsf{MS} \times [0,1]^\beta \twoheadrightarrow \mathsf{MS} \times [0,1]^\alpha$ is the natural projection.
(C3) $\pi^\beta_\alpha(Z_\beta) = Z_\alpha$ whenever $\alpha \leqslant \beta \leqslant \omega_1$.
(C4) Z_α is homeomorphic to MS whenever $\alpha < \omega_1$.
(C5) The maps $\sigma^\beta_\alpha := \pi^\beta_\alpha \restriction Z_\beta : Z_\beta \twoheadrightarrow Z_\alpha$, for $\alpha \leqslant \beta \leqslant \omega_1$, are monotone.

Using (C1)–(C3), the construction is determined at limit ordinals; (C4) is preserved by Lemma 2.5 and (C5). It remains to explain how, given Z_α for $\alpha < \omega_1$, we obtain $Z_{\alpha+1} \subseteq Z_\alpha \times [0,1]$; as usual, we identify $\mathsf{MS} \times [0,1]^{\alpha+1}$ with $\mathsf{MS} \times [0,1]^\alpha \times [0,1]$.

We now add:

Conditions 3.2. We have q^ξ_α and t_α for $\xi < \alpha < \omega_1$ such that:

(C6) Each $\langle q^\xi_\alpha\colon \xi < \alpha\rangle$ is a sequence of points in $\mathsf{MS} \times [0,1]^\alpha$.
(C7) Whenever $\langle q^\xi\colon \xi < \omega_1\rangle$ is any sequence of points in $\mathsf{MS} \times [0,1]^{\omega_1}$, $\{\alpha < \omega_1\colon \forall \xi < \alpha\ [\pi^{\omega_1}_\alpha(q^\xi) = q^\xi_\alpha]\}$ is stationary.
(C8) Whenever $\alpha < \beta \leqslant \omega_1$ and $z \in Z_\alpha$: If $q^\xi_\alpha \in Z_\alpha$ for all $\xi < \alpha$ and z is a limit point of $\{q^\xi_\alpha\colon \xi < \alpha\ \&\ q^\xi_\alpha \neq z\}$, then all points of $(\sigma^\beta_\alpha)^{-1}\{z\}$ are strong limit points of $\{(\sigma^\beta_\alpha)^{-1}\{q^\xi_\alpha\}\colon \xi < \alpha\}$.
(C9) $t_\alpha \in Z_\alpha$, and for all $z \in Z_\alpha$: $(\sigma^{\alpha+1}_\alpha)^{-1}\{z\}$ is a singleton if $z \neq t_\alpha$ and a non-locally-separating arc if $z = t_\alpha$.
(C10) $t_\alpha = q^0_\alpha$ whenever $\alpha > 0$ and $q^0_\alpha \in Z_\alpha$.

Proof of Theorem 1.1. The fact that one may obtain (C1)–(C10) has already been outlined above. (C6), (C7) are possible by $\diamondsuit$, and (C10) is just a definition. (C8), (C9) are obtained by induction on β. For the successor step, we must obtain $Z_{\beta+1}$ from Z_β using Lemmas 2.7 and 2.8. Here, $X = Z_\beta$, $Y = Z_{\beta+1}$, and $t = t_\beta$; the $\mathcal{F}_n$ list all sets of the form $\mathcal{F}^\beta_\alpha := \{(\sigma^\beta_\alpha)^{-1}\{q^\xi_\alpha\}\colon \xi < \alpha\ \&\ q^\xi_\alpha \in Z_\alpha\}$ such that $\alpha \leqslant \beta$ and t_β is a strong limit point of $\mathcal{F}^\beta_\alpha$. Observe that (C8) for $(\alpha, \beta+1)$ is immediate from (C8) for (α, β) *except* for the points of $Z_{\beta+1}$ in $(\sigma^{\beta+1}_\beta)^{-1}\{t_\beta\}$. Also observe that in order to apply Lemmas 2.7 and 2.8, we must check by induction on β, using Lemma 2.9, that the sets $(\sigma^\beta_\alpha)^{-1}\{q^\xi_\alpha\}$ are non-locally-separating (and nowhere dense) in Z_β.

Note that $\chi(z, Z) = \aleph_1$ for all $z \in Z$; this follows from (C9,C10) and the fact, using (C7), that $\{\alpha < \omega_1\colon \pi^{\omega_1}_\alpha(z) = t_\alpha\}$ is unbounded in ω_1.

Z is HS by (C6)–(C8), (C1)–(C3): If not, suppose that $\langle q^\xi\colon \xi < \omega_1\rangle$ is left-separated in Z. As in [5], we get a club $C \subset \omega_1$ such that for all $\alpha \in C$,

1. The $\sigma^{\omega_1}_\alpha(q^\xi)$ for $\xi < \alpha$ are all distinct; and
2. For all η with $\alpha \leqslant \eta < \omega_1$, $\sigma^{\omega_1}_\alpha(q^\eta)$ is a limit point of $\{\sigma^{\omega_1}_\alpha(q^\xi)\colon \xi < \alpha\}$.

Fix $\alpha \in C$ such that $\forall \xi < \alpha\,[\sigma^{\omega_1}_\alpha(q^\xi) = q^\xi_\alpha]$. Let $z = \sigma^{\omega_1}_\alpha(q^\alpha)$. Applying (C8) with $\beta = \omega_1$, we have in Z: all points of $(\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ are strong limit points of $\{(\sigma^{\omega_1}_\alpha)^{-1}\{q^\xi_\alpha\}\colon \xi < \alpha\}$. In particular, q^α is a limit point of $\langle q^\xi\colon \xi < \alpha\rangle$, contradicting "left-separated".

Similarly, Z has no nontrivial convergent sequences: Suppose that $q^n \to q^\omega$ in Z, where the q^ξ for $\xi \leqslant \omega$ are distinct. Let $q^\xi = q^\omega$ when $\omega < \xi < \omega_1$, and apply (C7) to get α with $\omega < \alpha < \omega_1$ such that the $\sigma^{\omega_1}_\alpha(q^\xi)$ for $\xi \leqslant \omega$ are distinct points and $\forall \xi < \alpha\ [\sigma^{\omega_1}_\alpha(q^\xi) = q^\xi_\alpha]$. Let $z = \sigma^{\omega_1}_\alpha(q^\omega)$. Then all points of $(\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ are strong limit points of $\{(\sigma^{\omega_1}_\alpha)^{-1}\{q^\xi_\alpha\}\colon \xi < \alpha\}$ and hence also of $\{(\sigma^{\omega_1}_\alpha)^{-1}\{q^n_\alpha\}\colon n < \omega\}$. So, all points of $(\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ are limit points of $\{q^n\colon n \in \omega\}$. Since $\{q^\omega\} \subsetneqq (\sigma^{\omega_1}_\alpha)^{-1}\{z\}$ (by $\chi(q^\omega, Z) = \aleph_1$), we contradict $q^n \to q^\omega$. $\square$

4. The almost clopen algebra

We show here (Theorem 4.12) that a space Z satisfying Theorem 1.1 cannot have a base of open sets with scattered boundaries; equivalently (because there are no nontrivial convergent sequences) with finite boundaries. We first note that if there were such a base, we could take the basic open sets U to be regular, since $\partial(\mathrm{int}(\mathrm{cl}(U))) \subseteq \partial U$. To simplify notation, we define:

Definition 4.1. $\mathrm{ro}(X)$ denotes the algebra of regular open subsets of X, and $\mathrm{acl}(X)$ (the *almost clopen* sets) denotes the family of regular open sets U such that ∂U is finite. For $U \in \mathrm{ro}(X)$, let $U^{\complement}$ denote the boolean complement $(X\backslash U)^\circ$.

Note that $\mathrm{acl}(X)$ is a boolean subalgebra of $\mathrm{ro}(X)$: If $U \in \mathrm{acl}(X)$ and $W = U^{\complement}$, then $\partial W = \partial U$, so $W \in \mathrm{acl}(X)$. Also, if $U, V \in \mathrm{acl}(X)$ and $W = U \wedge V = U \cap V \in \mathrm{ro}(X)$, then $W \in \mathrm{acl}(X)$ because $\partial(W) \subseteq \partial(U) \cup \partial(V)$.

In a locally connected space, the connected components of an open set U are open; if V is any such component, then $\partial V \subseteq \partial U$ (because V is relatively clopen in U), so $V \in \mathrm{acl}(X)$ whenever $U \in \mathrm{acl}(X)$. Thus,

Lemma 4.2. *If X is locally connected and $\mathrm{acl}(X)$ is a local base at $p \in X$, then $\{U \in \mathrm{acl}(X)\colon\ p \in U \ \&\ U \text{ is connected}\}$ is also a local base at p.*

Various LOTS sums have bases of almost clopen sets. This is true, for example, for any compact hedgehog consisting of a central point plus arbitrarily many LOTS spines. The assumption of no convergent sequences, however, puts some restrictions on the space. In particular, the hedgehog fails the following lemma (taking U to be X and letting s be the central point):

Lemma 4.3. *Assume that X is compact and locally connected, and X has no nontrivial convergent sequences. Fix an open U with ∂U finite, and fix a finite $s \subseteq U$. Then $U\backslash s$ has finitely many components.*

Proof. Assume that V_n, for $n < \omega$, are different components of $U\backslash s$. Choose $x_n \in V_n$. Then the limit points of $\{x_n\colon\ n \in \omega\}$ must lie in $\partial(U\backslash s) \subseteq \partial U \cup s$. Thus, $\{x_n\colon\ n \in \omega\}$ has finitely many limit points, which is impossible if X has no nontrivial convergent sequences. □

We now look more closely at the locally separating points; that is, the points $p \in X$ such that $U\backslash\{p\}$ is not connected for some open connected $U \ni p$.

Definition 4.4. If $p \in U \subseteq X$, then $c(p, U)$ is the number of components of $U\backslash\{p\}$.

Lemma 4.5. *Assume that X is compact and locally connected, and $p \in X$. If U and V are open connected subsets of X with $p \in V \subseteq U$, then*:

(1) *Every component of $V\backslash\{p\}$ is a subset of exactly one component of $U\backslash\{p\}$.*
(2) $c(p, V) \geqslant c(p, U)$.
(3) *If $\mathrm{acl}(X)$ is a local base at p and X has no nontrivial convergent sequences, then $c(p, U)$ is finite.*

Proof. (1) is immediate from the fact that if W is a component of $V\backslash\{p\}$ then W is connected and $W \subseteq U\backslash\{p\}$. For (2), use the fact that every component of $U\backslash\{p\}$ must meet V because U is connected, so that (1) provides a map from the components of $V\backslash\{p\}$ *onto* the components of $U\backslash\{p\}$. For (3), choose $V \in \mathrm{acl}(X)$ with $p \in V \subseteq U$, and apply (2) and Lemma 4.3. □

The next lemma is trivial, but useful when ∂U is finite.

Lemma 4.6. *Suppose that $E \subseteq X$ is connected, $U \subseteq X$ is open, and $\partial U \cap E = \emptyset$. Then $E \subseteq U$ or $E \cap U = \emptyset$.*

Proof. $U \cap E = \overline{U} \cap E$ is relatively clopen in E, so $U \cap E$ is either E or $\emptyset$. □

Lemma 4.7. *Assume that X is compact and locally connected, $\mathrm{acl}(X)$ is a local base at $p \in X$, and X has no nontrivial convergent sequences. Then there is an $n \in \omega$ such that $c(p, U) \leqslant n$ for all open connected $U \ni p$.*

Proof. If this fails, then applying Lemma 4.5, we may fix open connected $U_n \ni p$ for $n \in \omega$ such that $U_0 \supseteq \overline{U}_1 \supseteq U_1 \supseteq \overline{U}_2 \cdots$ and $2 \leqslant c(p, U_0) < c(p, U_1) < \cdots$. Then, we may define a subtree $T \subseteq \omega^{<\omega}$ and open connected W_s for $s \in T$ and $k_s \in \omega\backslash\{0\}$ for $s \in T$ as follows:

(1) $W_{()}$ is the component of p in X.
(2) If $\mathrm{lh}(s) = n$, then k_s is the number of components of $U_n\backslash\{p\}$ which are subsets of W_s, and these components are listed as $\{W_{s^\frown i}\colon\ i < k_s\}$.
(3) $s^\frown i \in T$ iff $s \in T$ and $i < k_s$.

Item (1) is a bit artificial, but it gives T a root node $()$. For the levels below the root, note that $|T \cap \omega^{n+1}| = c(p, U_n)$, and the W_s for $s \in T \cap \omega^{n+1}$ list the components of $U_n\backslash\{p\}$. Let $P(T) = \{f \in \omega^\omega\colon\ \forall n\ [f{\restriction}n \in T]\}$ be the set of paths through T.

Since every node in T has at least one child, $|P(T)|$ is either $\aleph_0$ or $2^{\aleph_0}$. Note that $\mathrm{cl}(W_{s^\frown i}) \subseteq W_s \cup \{p\}$, since if $n = \mathrm{lh}(s) > 0$ and $q \in \mathrm{cl}(W_{s^\frown i})\backslash\{p\}$, then q and the points of $W_{s^\frown i}$ must all lie in the same component of $U_{n-1}\backslash\{p\}$, which is W_s.

Let $H = \bigcap_n U_n = \bigcap_n \overline{U_n}$. Then H is a connected closed G_δ containing p, and H must be infinite, since p must have uncountable character. For each $f \in P(T)$, let $K_f = \bigcap_n \mathrm{cl}(W_{f\restriction n}) = \{p\} \cup \bigcap_n W_{f\restriction n}$. Then the K_f are connected and infinite, since $\{p\}$ cannot be a decreasing intersection of ω infinite closed sets (or there would be a convergent sequence). Observe that $K_f \cap K_g = \{p\}$ whenever $f \neq g$. Thus, if $p \in V \in \mathrm{acl}(X)$ then $K_f \subseteq V$ for all but finitely many $f \in P(T)$, since $K_f \subseteq V$ whenever $K_f \cap \partial V = \emptyset$ by Lemma 4.6. Now let f_i, for $i \in \omega$ be distinct elements of $P(T)$, and choose $q_i \in K_{f_i}\backslash\{p\}$. Then every neighborhood of p contains all but finitely many q_i, so the q_i converge to p, a contradiction. $\square$

Definition 4.8. Assume that X is compact and locally connected, $\mathrm{acl}(X)$ is a base for X, and X has no nontrivial convergent sequences. Then for each $p \in X$, define $c(p) \in \omega$ to be the largest $c(p, U)$ among all open connected $U \ni p$.

By a standard chaining argument:

Lemma 4.9. *Assume that X is compact and locally connected and $\mathrm{acl}(X)$ is a base for X. Fix a connected open $U \subseteq X$ and a compact $F \subseteq U$. Then there is a connected $V \in \mathrm{acl}(X)$ such that $F \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Let $\mathcal{G} = \{W \in \mathrm{acl}(X)\colon\ \emptyset \neq \overline{W} \subseteq U\ \&\ W$ is connected$\}$. Then $\bigcup \mathcal{G} = U$. View $\mathcal{G}$ as an undirected graph, by putting an edge between W_1 and W_2 iff $W_1 \cap W_2 \neq \emptyset$. Then $\mathcal{G}$ is connected as a graph because U is connected and the components of $\mathcal{G}$ yield topological components of U. Fix a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $F \subseteq \bigcup \mathcal{G}_0$. Then fix a finite connected $\mathcal{G}_1$ with $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}$. Let $V = \bigvee \mathcal{G}_1 = \mathrm{int}(\mathrm{cl}(\bigcup \mathcal{G}_1))$. $\square$

Lemma 4.10. *Assume that X is compact and locally connected, $\mathrm{acl}(X)$ is a base for X, and X has no nontrivial convergent sequences. Then there is no sequence of open sets $\langle U_n\colon\ n \in \omega\rangle$ such that $\overline{U}_{n+1} \subsetneqq U_n$ for all n and $\overline{U}_n\backslash U_{n+1}$ is connected for all* even *n.*

Proof. Given such a sequence, choose $x_n \in \overline{U}_n\backslash U_{n+1}$, and let y be a limit point of $\{x_{2m}\colon\ m \in \omega\}$. Since $\langle x_{2m}\colon\ m \in \omega\rangle$ cannot converge to y, fix a connected $W \in \mathrm{acl}(X)$ and disjoint infinite $A, B \subseteq \{2m\colon\ m \in \omega\}$ such that $x_n \in W$ for all $n \in A$ and $x_n \notin W$ for all $n \in B$. Since ∂W is finite, we may also assume (shrinking A and B if necessary) that $\partial W \cap (\overline{U}_n\backslash U_{n+1}) = \emptyset$ for all $n \in A \cup B$. Then, by Lemma 4.6, $\overline{U}_n\backslash U_{n+1} \subseteq W$ for all $n \in A$ and $(\overline{U}_n\backslash U_{n+1}) \cap W = \emptyset$ for all $n \in B$. But then, for $n \in B$, the connected W is partitioned into the disjoint open sets $W \cap U_{n+1}$, $W \setminus \overline{U_n}$, both of which are non-empty when $n > \min(A)$. $\square$

Lemma 4.11. *Assume that X is compact and locally connected, $\mathrm{acl}(X)$ is a base for X, and X has no nontrivial convergent sequences. Then every non-isolated point in X is locally separating.*

Proof. Suppose we have a non-isolated p which is not locally separating; so $U\backslash\{p\}$ is connected whenever U is open and connected. Then recursively construct U_n for $n \in \omega$ such that:

(1) Each U_n is open and $p \in U_n$.
(2) Each $\overline{U}_{n+1} \subsetneqq U_n$.
(3) $\overline{U}_n\backslash U_{n+1}$ is connected whenever n is even.
(4) Each $U_n \in \mathrm{acl}(X)$.
(5) U_n is connected for all even n.

Then (1)–(3) contradict Lemma 4.10.

To construct the U_n: Let $U_0 \in \mathrm{acl}(X)$ be such that $p \in U_0$ and U_0 is connected and not clopen. Given U_n, where n is even, we construct U_{n+1} and U_{n+2} as follows.

Say $\partial U_n = \{q^j\colon\ j < r\}$; of course, r and the q^j depend on n. For each j, choose $V^j \in \mathrm{acl}(X)$ be such that $q^j \in V^j$, $p \notin \mathrm{cl}(V^j)$, and V^j is connected. Also make sure that the $\overline{V^j}$ are disjoint; then $\overline{V^j} \cap \partial U_n = \{q^j\}$. Let $\{W_i^j\colon\ i < c^j\}$ list the components of $V^j\backslash\{q^j\}$; so $2 \leqslant c^j < \omega$. Then W_i^j is connected and $\partial U_n \cap W_i^j = \emptyset$, so $W_i^j \subseteq U_n$ or $W_i^j \cap U_n = \emptyset$; say $W_i^j \subseteq U_n$ for $i < d^j$ and $W_i^j \cap U_n = \emptyset$ for $d^j \leqslant i < c^j$; so $1 \leqslant d^j < c^j$. Choose $y_i^j \in W_i^j$. Now U_n is connected and p is not locally separating, so $U_n\backslash\{p\}$ is connected. Applying Lemma 4.9, fix a connected $R \in \mathrm{acl}(X)$ such that $\{y_i^j\colon\ j < r\ \&\ i < d^j\} \subseteq R \subseteq \overline{R} \subseteq U\backslash\{p\}$. Let S be the finite union $R \cup \bigcup\{W_i^j\colon\ j < r\ \&\ i < d^j\}$. Then S is open and connected, $p \notin \overline{S}$, and each $q^j \in \overline{S}$. Let $U_{n+1} = U_n\backslash\overline{S} = \overline{U_n} \setminus \overline{S}$. Then $p \in U_{n+1} \in \mathrm{acl}(X)$, and $\overline{U}_n\backslash U_{n+1} = \overline{S}$ is connected. Also, each $q^j \notin \overline{U_{n+1}}$ because $U_{n+1} \cap V^j = \emptyset$, so that $\overline{U_{n+1}} \subseteq U_n$.

Now, choose a connected $U_{n+2} \in \mathrm{acl}(X)$ so that $p \in U_{n+2} \subseteq \overline{U_{n+2}} \subsetneqq U_{n+1}$. $\square$

Theorem 4.12. *If X is infinite, compact, locally connected, and $\mathrm{acl}(X)$ is a base for X, then X has a nontrivial convergent sequence.*

Proof. Suppose not. Fix any non-isolated $p \in X$; then p is locally separating by Lemma 4.11, so $c(p) \geqslant 2$ (see Definition 4.8). Fix a connected $U \in \mathrm{acl}(X)$ such that $p \in U$ and $c(p, U) = c(p)$. Let W_i, for $i < c(p)$ be the components of $U \backslash \{p\}$. Then $c(p, V) = c(p)$ whenever $V \in \mathrm{acl}(X)$ and $p \in V \subseteq U$; furthermore, the components of $V \backslash \{p\}$ are the sets $W_i \cap V$ for $i < c(p)$.

Let $Y = \mathrm{cl}(W_0)$. Then $\mathrm{acl}(Y)$ is a base for Y, Y is locally connected, and Y has no nontrivial convergent sequences. Furthermore, $p \in Y$ and p is not locally separating in Y, contradicting Lemma 4.11 applied to Y. $\square$

5. Strong S-spaces of various dimensions

Call Z a *Fedorchuk space* iff Z is compact HS and crowded, and has no nontrivial convergent sequences. So, Theorem 1.1 produces, under $\diamondsuit$, a one-dimensional locally connected Fedorchuk space. Using the same method, one can modify the CH construction of van Mill [15] to produce, under $\diamondsuit$, an infinite dimensional locally connected Fedorchuk space; in this construction, the Hilbert cube replaces the Menger sponge MS. The $\diamondsuit$ is necessary since by Eisworth [6], CH alone does not imply the existence of any Fedorchuk space.

The referee of the original version of this paper asked whether one might also produce a k-dimensional locally connected Fedorchuk space for each finite $k \geqslant 1$. One way of doing this (the referee's suggestion) is to replace MS by Menger's universal k-dimensional compactum; these spaces are described in detail in Bestvina [3]. We are not sure if this works, since the characterization of these compacta for $k > 1$ is a bit more complex than that for MS. However, we can construct our Z so that the product Z^k provides a k-dimensional example.

Let Z be as constructed in our proof of Theorem 1.1. Then $\dim(Z^k) = k$ because Z^k is an inverse limit of copies of MS^k, which has dimension k. Also, Z^k is certainly crowded and locally connected, and has no nontrivial convergent sequences. We need to do some extra work to ensure that Z^k is HS for all $k < \omega$; that is, Z is a *strong* S-space. Then Z^ω will also be HS, but Z^ω has nontrivial convergent sequences.

The key to making our space HS was conditions (C6)–(C8), where we used $\diamondsuit$ to capture all ω_1-sequences from Z, ensuring that no such sequence is left-separated. But we can also use $\diamondsuit$ to capture sequences from Z^k, which in our construction is a subspace of $(\mathrm{MS} \times [0,1]^{\omega_1})^k$. To avoid confusion in our subscripts, if $y \in Y^k$, let ${}_\mu y$, for $\mu < k$, denote coordinate μ of y. Call a point $y \in Y^k$ *simple* iff all the ${}_\mu y$ are different, and call a γ-sequence $\langle q^\xi \colon \xi < \gamma\rangle$ from Y^k *simple* iff ${}_\mu q^\xi \neq {}_\nu q^\eta$ unless $\mu = \nu$ and $\xi = \eta$. Observe that for Z to be strongly HS, it is sufficient that for each k, there are no simple left-separated ω_1-sequences in Z^k.

To avoid confusion about which k is handled at each stage, partition ω_1 into disjoint stationary sets S_k for $k < \omega$ such that $\diamondsuit(S_k)$ is true for each k. In (C7), require that $\{\alpha \in S_1\colon \forall \xi < \alpha\, [\pi^{\omega_1}_\alpha(q^\xi) = q^\xi_\alpha]\}$ be stationary; then Z is HS and has no convergent ω-sequences. To make Z^k HS, add the following when $2 \leqslant k < \omega$:

(C6^k) For $\alpha \in S_k$, $\langle q^\xi_\alpha \colon \xi < \alpha\rangle$ is a simple sequence of points in $(\mathrm{MS} \times [0,1]^\alpha)^k$.

(C7^k) Whenever $\langle q^\xi \colon \xi < \omega_1\rangle$ is any simple sequence of points in $(\mathrm{MS} \times [0,1]^{\omega_1})^k$, $\{\alpha \in S_k\colon \forall \xi < \alpha\ [\pi^{\omega_1}_\alpha(q^\xi) = q^\xi_\alpha]\}$ is stationary.

(C8^k) Whenever $\alpha \in S_k$ and $\alpha < \beta \leqslant \omega_1$ and $z \in (Z_\alpha)^k$: If $q^\xi_\alpha \in (Z_\alpha)^k$ for all $\xi < \alpha$ and z is a limit point of $\{q^\xi_\alpha \colon \xi < \alpha$ & $q^\xi_\alpha \neq z\}$, then all points of $(\sigma^\beta_\alpha)^{-1}\{z\}$ are strong limit points of $\{(\sigma^\beta_\alpha)^{-1}\{q^\xi_\alpha\} \colon \xi < \alpha\}$.

Here, π^β_α denotes the natural projection from $(\mathrm{MS} \times [0,1]^\beta)^k$ onto $(\mathrm{MS} \times [0,1]^\alpha)^k$, and σ^β_α denotes the natural projection from $(Z_\beta)^k$ onto $(Z_\alpha)^k$.

Then, to achieve (C8^k), we need the following improvement on Lemma 2.7. Call a nonempty $F \subseteq X^k$ a *nice closed k-box* iff $F = \prod_{\mu<k}({}_\mu F)$, where each ${}_\mu F$ is closed and not locally separating in X, and the ${}_\mu F$ are pairwise disjoint; then write $\mathrm{Sides}(F)$ for $\bigcup_{\mu<k}({}_\mu F)$. Call $\mathcal{F}$ a *nice k-family* iff $|\mathcal{F}| = \aleph_0$ and each $F \in \mathcal{F}$ is a nice closed k-box and $\mathrm{Sides}(F) \cap \mathrm{Sides}(\widetilde{F}) = \emptyset$ whenever $F, \widetilde{F}$ are distinct elements of $\mathcal{F}$. Call $\mathcal{F}$ a *nice family* iff $\mathcal{F}$ is a nice k-family for some k with $0 < k < \omega$.

Lemma 5.1. *Suppose that $X \cong \mathrm{MS}$ and $\mathfrak{F}$ is a countable set of nice families. Fix any $t \in X$. Then there is a $Y \cong \mathrm{MS}$ and a monotone $\sigma : Y \twoheadrightarrow X$ such that*:

1. *$\sigma^{-1}\{t\}$ is a non-locally-separating arc in Y,*
2. *$|\sigma^{-1}\{x\}| = 1$ for all $x \neq t$, and*
3. *For each $k \in \omega$ and $y \in Y^k$, if $\sigma(y)$ is a strong limit point of a k-family $\mathcal{F} \in \mathfrak{F}$, then y is a strong limit point of $\{\sigma^{-1}(F)\colon F \in \mathcal{F}\}$. Here, σ is applied to each coordinate of y; likewise, σ^{-1} operates coordinatewise.*

When $k = 1$, the result is trivial when $\sigma(y) \neq t$, and Lemma 2.7 handles those y for which $\sigma(y) = t$. Lemma 2.7 did not require the sets in $\mathcal{F}$ to be disjoint, but they are disjoint when the lemma is applied to the proof that Z is HS, since our $\mathcal{F}$ arises from an inverse limit of a simple sequence. When $k > 1$, we cannot assume that y is simple, so we must consider the possibility that $\sigma({}_\mu y) = t$ for some μ and not for other μ.

Proof of Lemma 5.1. For each nice k-family $\mathcal{F}$, we describe some related families as follows: Fix r with $1 \leqslant r \leqslant k$, fix $Q = \{\mu_0, \ldots, \mu_{r-1}\}$ with $\mu_0 < \cdots < \mu_{r-1} < k$, and fix a $(k-r)$-tuple $\vec{V} = \langle {}_\mu V\colon\ \mu \in k\backslash Q\rangle$ of basic open subsets of X. Let $\mathcal{F}\restriction(Q, \vec{V})$ be the family of all nice closed r-boxes H such that for some $F \in \mathcal{F}$: ${}_\nu H = {}_{\mu_\nu} F$ for $\nu < r$ and ${}_\mu F \subseteq {}_\mu V$ for $\mu \in k\backslash Q$. Note that $\mathcal{F}\restriction(Q, \vec{V})$ is a nice k-family unless it is finite. If $r = k$, then $Q = k$ and $\vec{V}$ is the empty sequence and $\mathcal{F}\restriction(Q, \vec{V}) = \mathcal{F}$; this will handle the special case where all $\sigma({}_\mu y) = t$.

Call t a *sidewise strong limit* of a nice k-family $\mathcal{F}$ iff for all open $U \ni t$, $\mathrm{Sides}(F) \subseteq U$ for all but finitely many $F \in \mathcal{F}$.

Observe that we may assume the following closure properties of $\mathfrak{F}$:

(a) If $\mathcal{F} \in \mathfrak{F}$ and t is a sidewise strong limit of some infinite $\widetilde{\mathcal{F}} \subseteq \mathcal{F}$, then some such $\widetilde{\mathcal{F}}$ is in $\mathfrak{F}$.
(b) If $\mathcal{F} \in \mathfrak{F}$ and $Q, \vec{V}$ are as above, then $\mathcal{F}\restriction(Q, \vec{V}) \in \mathfrak{F}$ unless $\mathcal{F}\restriction(Q, \vec{V})$ is finite.

We next restate that part of the proof of Lemma 2.7 which remains unchanged here.

In X, we shall choose disjoint closed non-locally-separating $D_i \not\ni t$ for $i \in \omega$ such that every neighborhood of t contains all but finitely many of the D_i. Let $L = \{t\} \cup \bigcup_i D_i$. Then L is closed and non-locally-separating.

In MS, let J be any non-locally-separating arc. We shall choose disjoint closed non-locally separating sets G_i for $i \in \omega$ such that each $G_i \cong D_i$ and every neighborhood of J contains all but finitely many G_i.

$\rho : \mathrm{MS} \twoheadrightarrow \mathrm{MS}/J$ is the usual projection. Then $\mathrm{MS}/J \cong \mathrm{MS}$. In MS/J, let $K = \{[J]\} \cup \bigcup\{\rho(G_i)\colon\ i \in \omega\}$. Let $h : K \twoheadrightarrow L$ be a homeomorphism such that $h([J]) = t$ and each $h(\rho(G_i)) = D_i$; then h extends to a homeomorphism $\widetilde{h} : \mathrm{MS}/J \twoheadrightarrow X$. Let $Y = \mathrm{MS}$ and let $\sigma = \widetilde{h} \circ \rho$. This handles everything in Lemma 5.1 except for (3), which requires more about the D_i and G_i.

In addition to the preceding requirements, choose the D_i and G_i so that for all basic open ${}_0U, \ldots, {}_{k-1}U \subseteq \mathrm{MS}$ which meet J: whenever t is a sidewise strong limit of a k-family $\mathcal{F} \in \mathfrak{F}$, there are infinitely many $n \in \omega$ such that for some $F \in \mathcal{F}$ and all $\mu < k$: $D_{n+\mu} = {}_\mu F$ and $G_{n+\mu} \subseteq {}_\mu U \backslash J$.

To see that this proves Lemma 5.1: Fix any k-family $\mathcal{F} \in \mathfrak{F}$. Fix any $y \in Y^k$, let $x = \sigma(y) \in X^k$, and assume that x is a strong limit point of $\mathcal{F}$. We need to show that y is a strong limit point of $\{\sigma^{-1}(F)\colon\ F \in \mathcal{F}\}$. Assume that exactly r of the coordinates of x equal t. Since the result is trivial if $r = 0$, assume that $1 \leqslant r \leqslant k$. Let $Q = \{\mu_0, \ldots, \mu_{r-1}\}$, with $\mu_0 < \cdots < \mu_{r-1} < k$, list the subscripts μ with ${}_\mu x = t$.

Fix basic open neighborhoods ${}_\mu U \ni {}_\mu y$ for $\mu < k$; when $\mu \notin Q$, assume that ${}_\mu U \cap J = \emptyset$ and ${}_\mu U = \sigma^{-1}({}_\mu V)$, where ${}_\mu V$ is a basic open neighborhood of ${}_\mu x$ in X. This defines $\vec{V}$. Since x is a strong limit point of $\mathcal{F}$, $\mathcal{F}\restriction(Q, \vec{V})$ is infinite, so $\mathcal{F}\restriction(Q, \vec{V}) \in \mathfrak{F}$ and hence t is a sidewise strong limit of some infinite $\widetilde{\mathcal{F}} \subseteq \mathcal{F}\restriction(Q, \vec{V})$ which, by closure property (a), is in $\mathfrak{F}$. Then there are infinitely many n such that for some $H \in \widetilde{\mathcal{F}}$ and all $\nu < r$: $D_{n+\nu} = {}_\nu H$ and $G_{n+\nu} \subseteq {}_{\mu_\nu} U$. For these H, there is an $F \in \mathcal{F}$ such that ${}_\mu F \subseteq {}_\mu V$ for $\mu \notin Q$ and each ${}_{\mu_\nu} F = {}_\nu H$; then $\sigma^{-1}({}_\mu F) \subseteq {}_\mu U$ for all μ. There are thus infinitely many $F \in \mathcal{F}$ with $\sigma^{-1}({}_\mu F) \subseteq {}_\mu U$ for all μ. □

6. Further remarks

We note that in constructing a locally connected compactum, the monotone bonding maps, as used also by van Mill [15], are inevitable.

Remark 6.1. Assume that $X \subseteq [0,1]^{\omega_1}$ is compact and locally connected. Define $X_\alpha = \pi_\alpha^{\omega_1}(X) \subseteq [0,1]^\alpha$. Then there is a club $C \subseteq \omega_1$ such that X_α is locally connected for all $\alpha \in C$, and such that $\sigma_\alpha^\beta := \pi_\alpha^\beta \restriction X_\beta$ is monotone whenever $\alpha < \beta$ and $\alpha, \beta \in C \cup \{\omega_1\}$.

Proof. Let $\mathcal{B}$ be the family of all connected open F_σ subsets of X. Then $\mathcal{B}$ is a base for X. For $\alpha < \omega_1$, let $\mathcal{B}_\alpha$ be the family of all open $U \subseteq X_\alpha$ such that $(\sigma_\alpha^{\omega_1})^{-1}(U) \in \mathcal{B}$. Observe that each $U \in \mathcal{B}_\alpha$ is connected. Put $\alpha \in C$ iff $\mathcal{B}_\alpha$ is a base for X_α. Then C is club.

Now, it is sufficient to show that $(\sigma_\alpha^{\omega_1})^{-1}\{x\}$ is connected whenever $\alpha \in C$ and $x \in X_\alpha$. Choose $U_n \in \mathcal{B}_\alpha$ with $x \in U_n \supseteq \overline{U_{n+1}}$ for all $n \in \omega$ and $\{x\} = \bigcap_n U_n = \bigcap_n \overline{U_n}$. Each $(\sigma_\alpha^{\omega_1})^{-1}(U_n)$ is in $\mathcal{B}$, so it and its closure are connected, and $\mathrm{cl}((\sigma_\alpha^{\omega_1})^{-1}(U_{n+1})) \subseteq (\sigma_\alpha^{\omega_1})^{-1}(\overline{U_{n+1}}) \subseteq (\sigma_\alpha^{\omega_1})^{-1}(U_n)$, so that $(\sigma_\alpha^{\omega_1})^{-1}\{x\}$ is the decreasing intersection of the connected closed sets $\mathrm{cl}((\sigma_\alpha^{\omega_1})^{-1}(U_n))$, and is hence connected. □

We do not know if conditions (C1)–(C10) in Section 3 determine $\mathrm{ind}(Z)$, but a minor addition to the construction will ensure that Z does not have small *transfinite inductive dimension*; that is, $\mathrm{trind}(Z) = \infty$ (and hence $\mathrm{ind}(Z) = \infty$). The transfinite inductive dimension trind is the natural generalization of ind; see [7].

Theorem 6.2. *Assuming $\diamondsuit$, there is a locally connected HS continuum Z such that $\dim(Z) = 1$, $\mathrm{trind}(Z) = \infty$, and Z has no nontrivial convergent sequences.*

To do this, we make sure that all perfect subsets are G_δ sets. Observe that by local connectedness, every non-empty closed G_δ contains a non-empty connected closed G_δ subset, which in our Z cannot be a singleton. So, no non-empty closed G_δ can have dimension 0.

Lemma 6.3. *Assume that X is compact, connected, and infinite, and all perfect subsets of X are G_δ sets. Assume also that $\chi(x, X) > \aleph_0$ for all $x \in X$, and that in X, every non-empty closed G_δ set contains a non-empty closed connected G_δ subset. Then* $\mathrm{trind}(X) = \infty$.

Proof. We prove by induction on ordinals α that $\neg[\mathrm{trind}(X) \leqslant \alpha]$ for all such X. This is obvious for $\alpha = 0$. Assume $\alpha > 0$ and the inductive hypothesis holds for all ordinals $\xi < \alpha$. Suppose that $\mathrm{trind}(X) \leqslant \alpha$. Then there is a regular open set U such that $U \neq \emptyset$, $U \neq X$, and $\mathrm{trind}(\partial U) = \xi < \alpha$. Let $V = X \backslash \overline{U}$; then $\overline{U}$ and $\overline{V}$ are perfect, so $\partial U = \overline{U} \cap \overline{V}$ is a G_δ, and hence contains a non-empty closed connected G_δ subset Y. Then $\mathrm{trind}(Y) \leqslant \mathrm{trind}(\partial U) \leqslant \xi$. Since Y satisfies the conditions of the lemma, this is a contradiction. $\square$

By the same argument, this space is *weird* in the sense of [10]; that is, no perfect subset is totally disconnected.

To construct our Z so that perfect sets are G_δ, we observe first that if $Q \subseteq \mathsf{MS} \times [0,1]^{\omega_1}$ is perfect, then $C := \{\alpha < \omega_1\colon \pi^{\omega_1}_\alpha(Q) \text{ is perfect}\}$ is a club. One might then use $\diamondsuit$, as in [5], to capture perfect subsets of Z, but this is not necessary, since we already know that Z is HS, and we are already capturing countable sequences. Thus, we get:

Conditions 6.4. We have P_α and $\mathcal{P}_\alpha$ for $\alpha < \omega_1$ such that:

(C11) $P_\alpha = \mathrm{cl}(Z_\alpha \cap \{q^n_\alpha\colon n \in \omega\})$ whenever $\alpha \geqslant \omega$ and this set is perfect; otherwise, $P_\alpha = Z_\alpha$.
(C12) $\mathcal{P}_\alpha = \{(\sigma^\alpha_\delta)^{-1}(P_\delta)\colon \delta \leqslant \alpha\}$.
(C13) $\sigma^{\alpha+1}_\alpha \restriction ((\sigma^{\alpha+1}_\alpha)^{-1}(P)) : (\sigma^{\alpha+1}_\alpha)^{-1}(P) \twoheadrightarrow P$ is irreducible for each $P \in \mathcal{P}_\alpha$.

Proof of Theorem 6.2. To obtain these conditions, note that (C13) is trivial for P unless $t_\alpha \in P$. If $t_\alpha \in P$, then, since P is perfect, we may choose a sequence of distinct points $\langle p_n\colon n \in \omega\rangle$ from $P \backslash \{t_\alpha\}$ converging to t_α. Then, while we are accomplishing (C8), we make sure that all points of $(\sigma^{\alpha+1}_\alpha)^{-1}\{t_\alpha\}$ are (strong) limit points of the set of singletons, $\{(\sigma^{\alpha+1}_\alpha)^{-1}\{p_n\}\colon n \in \omega\}$; this implies irreducibility.

Now, we prove by induction on $\beta \geqslant \alpha$ that $\sigma^\beta_\alpha \restriction ((\sigma^\beta_\alpha)^{-1}(P)) : (\sigma^\beta_\alpha)^{-1}(P) \twoheadrightarrow P$ is irreducible for each $P \in \mathcal{P}_\alpha$. Then, if $Q \subseteq Z$ is perfect, we use HS and (C7) to fix some $\alpha < \omega_1$ such that $P_\alpha = \sigma^{\omega_1}_\alpha(Q)$ and P_α is perfect. Irreducibility then implies that $Q = (\sigma^{\omega_1}_\alpha)^{-1}(P_\alpha)$, which is a G_δ. $\square$

Finally, we remark that our space Z is *dissipated* in the sense of [12], since in the inverse limit, only one point t_α gets expanded in passing from Z_α to $Z_{\alpha+1}$; the inverse projection of every other point is a singleton. As pointed out in [12], this is also true of the original Fedorchuk S-space [8], where one point t_α got expanded to a pair of points; here, and in [10] and van Mill [15], t_α gets expanded to an interval.

Acknowledgements

The authors wish to thank the referee for a number of useful comments on the original version of this paper, and for raising the question which led to Section 5.

References

[1] R.D. Anderson, A characterization of the universal curve and a proof of its homogeneity, Ann. of Math. (2) 67 (1958) 313–324.
[2] R.D. Anderson, One-dimensional continuous curves and a homogeneity theorem, Ann. of Math. (2) 68 (1958) 1–16.
[3] M. Bestvina, Characterizing k-dimensional universal Menger compacta, Mem. Amer. Math. Soc. 71 (380) (1988) vi+110 pp.
[4] E.K. van Douwen, W.G. Fleissner, Definable forcing axiom: An alternative to Martin's axiom, Topology Appl. 35 (2–3) (1990) 277–289.
[5] M. Džamonja, K. Kunen, Measures on compact HS spaces, Fund. Math. 143 (1993) 41–54.
[6] T. Eisworth, Countable compactness, hereditary π-character, and the continuum hypothesis, Topology Appl. 153 (2006) 3572–3597.
[7] R. Engelking, Theory of Dimensions Finite and Infinite, Heldermann Verlag, Lemgo, 1995.
[8] V.V. Fedorčuk, The cardinality of hereditarily separable bicompacta, Dokl. Akad. Nauk SSSR 222 (1975) 302–305 (in Russian); English translation: Soviet Math. Dokl. 16 (1975) 651–655.
[9] V.V. Fedorchuk, A compact space having the cardinality of the continuum with no convergent sequences, Math. Proc. Camb. Phil. Soc. 81 (1977) 177–181.
[10] J. Hart, K. Kunen, Inverse limits and function algebras, Topology Proc. 30 (2) (2006) 501–521.
[11] J.R. Kline, Concerning the complement of a countable infinity of point sets of a certain type, Bull. Amer. Math. Soc. 23 (1917) 290–292.
[12] K. Kunen, Dissipated compacta, Topology Appl. 155 (2008) 282–303.
[13] K. Menger, Kurventheorie, Teubner, Leipzig, 1932.
[14] J.C. Mayer, L.G. Oversteegen, E.D. Tymchatyn, The Menger curve. Characterization and extension of homeomorphisms of non-locally-separating closed subsets, Dissertationes Math. (Rozprawy Mat.) 252 (1986), 45 pp.
[15] J. van Mill, A locally connected continuum without convergent sequences, Topology Appl. 126 (1–2) (2002) 273–280.
[16] J. Prajs, A homogeneous arcwise connected non-locally-connected curve, Amer. J. Math. 124 (2002) 649–675.